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Analysis of strength data using two- and three-parameter Weibull models

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The strength of ceramic components is adequately described using the Weibull distribution. Values of Weibull modulus and characteristic strength were obtained for two- and three-parameter models for chemically toughened glass and phosphate-bonded investment. The data for glass consisted of strength measurements from fifty specimens loaded in a similar way to operation. The data for phosphate-bonded investment was obtained by measuring four-point bend strength of fifty specimens at room temperature. All values of strength were plotted on modified Weibull probability scales on the *x*-axis against median ranks on the *y*-axis. The Weibull modulus for the two-parameter model was determined from the slope of the best fit straight line through the data. For the three-parameter model a correction was made to the "curve" of the two-parameter model by determining the probable value of the lower bound of strength. Ranks were also used to establish tolerance limits for design strengths.

The values of Weibull modulus obtained from the two-parameter and three-parameter models were similar for chemically toughened glass but different for phosphate-bonded investment. The three-parameter Weibull distribution was found to give a more reliable estimate of the Weibull modulus for phosphate-bonded investment.

1. Introduction

The strength of ceramic components is an important design criterion that depends in part on size and shape but often more importantly on surface or internal flaw size and distribution as well as the inherent cohesive strength of the material. Phosphate-bonded investment materials are a class of ceramic material that are currently used in the production of heat-resistant moulds for dental castings in cobalt–chromium alloys. The set material consists of particles of silica refractory in a matrix of magnesium ammonium phosphate. Phosphate-bonded investment materials have also been used more recently for the fabrication of dies for the superplastic forming of dental implant superstructures [1] and denture bases [2–10] in titanium alloy. In the superplastic forming process dies of phosphate-bonded investment are produced from a slurry of powder and water and poured into moulds. The set material is then removed from the mould and inserted into the tool of the metal forming press. At temperatures around 900 °C argon gas is used to force the titanium alloy sheet down onto the surface of the die and the detail of the die surface is transferred to the metal sheet. Thus it is possible to transfer the detailed undulations of the mucosa into the metal plate to make accurate fit surfaces for metal denture bases.

Changes in the use of investment materials such as the one described above necessarily require increased knowledge about this class of materials as the environment in which the metal forming operation occurs is somewhat different from the situation in which molten metal is cast into refractory moulds. The investment material is heated to similar temperatures both for casting and metal forming – the cast mould is heated to 950 °C even though the molten metal is around 1350 °C when it enters the mould in the casting process. However, the stress fields are not equivalent. In the metal forming process the die is essentially in compression although a degree of bending or point loading will occur. One of the limitations of die design will be its ability to withstand these stresses. Surface and internal flaws may well play a part in the materials' resistance to fracture and an analysis of strength data is required for this design process.

The Weibull distribution has been shown to be an appropriate model to describe strength data for ceramic materials [12] and two- and three-parameter Weibull models have been adopted to do so [13]. It is the purpose of this study to apply Weibull analysis to two data sets generated from two different ceramic materials and to test the accuracy of the model to describe the data. The first data set of chemically toughened glass is used for comparison in that the data set was taken from the literature [12] and has the same large sample size as that produced for phosphate-bonded investment which was generated by the method used for the manufacture of superplastic forming dies.

2. Materials and methods

Two data sets each comprising of fifty measurements of fracture strength were acquired. The data set for chemically toughened glass was obtained from the literature [12] whilst that for phosphate-bonded investment was produced by mixing in air by hand at a water-to-powder ratio of 0.12. The specimens of phosphate-bonded investment were produced by pouring the slurry into a mould of polyvinyl siloxane duplicating material. The set specimens had the dimensions 100 mm length × 15 mm width × 15 mm height. Measurements of fracture strength were made using a four-point bend test configuration on an Instron tensile testing machine and specimens were deformed at a rate of 1 mm min^{-1}.

The fifty measurements of fracture strength from each group were ranked in order from least strong to most strong. In each case the data set was divided into six groups and histograms drawn showing frequency of data points in each group. However, because often only small samples are available the shape of these histograms can vary significantly with change in class size. Therefore, a cumulative distribution plot is often preferred.

Thus, both data sets were transformed according to the following equations to plot the data on scales equivalent to Weibull probability paper to obtain values of the two- and three-parameter models for both materials [13].

2.1. Two-parameter model

1. Arrange strength values in increasing order.
2. Assign median ranks to these values of strength using the following approximation

$$Median\ rank = \left(\frac{i^{-0.3}}{n^{+0.4}}\right) \quad (1)$$

where i = failure order number, n = sample size.

3. Plot the data on scales equivalent to Weibull probability paper with strength on the abcissa and median ranks on the ordinate

$$Y = \ln\ln\left(\frac{1}{1 - Median\ rank}\right) \quad (2)$$

$$X = \ln(strength) \quad (3)$$

4. Draw the best fitting line through the points (fit by least squares method).
5. The slope of the line is the Weibull modulus, m.
6. The characteristic strength, θ, is the strength corresponding to 63.2% failed:

$$\theta = \exp\left(-\left(\frac{intercept}{m}\right)\right) \quad (4)$$

The cumulative distribution function shown below is then plotted for the data

$$CDF = 1 - \exp\left\{-\left[\left(\frac{x}{\theta}\right)^m\right]\right\} \quad (5)$$

where x = strength (MPa).

2.2. Three-parameter model

1. Arrange strength values in increasing order.
2. Assign median ranks to these values of strength using Equation 1.
3. Plot the data on scales equivalent to Weibull probability paper with strength on the abcissa and median ranks on the ordinate (Equations 2 and 3).
4. Draw the best fitting line through the points (fit by least squares method).
5. A correction is made to the resultant curve by determining the probable value of the lower bound of strength x_0. This value should be somewhere between the lowest measured value of the sample and zero. By trial x_0 is subtracted from the original set of data and plotted against median ranks (Y) as before.
6. By the least squares method a line is fitted to the modified data set.
7. A range of values of x_0 are selected and the data modified and plotted as above until the value of x_0 corresponding to the line with best fit is identified.
8. The characteristic strength can then be computed from the relationship

$$\theta = \exp\left[-\left(\frac{intercept}{m}\right)\right] + x_0 \quad (6)$$

9. The cumulative distribution function for the three-parameter model can then be plotted for the data

$$CDF = 1 - \exp\left\{-\left[\left(\frac{x - x_0}{\theta - x_0}\right)^m\right]\right\} \quad (7)$$

where x = strength(MPa).

2.3. Determination of tolerance limits

To determine tolerance limits for the jth value in n, the following procedure is appropriate [14]. The true rank is given by ${}_nZ_j$ and is to be estimated as it is an unknown. It is necessary to determine the true rank of the jth observation in n. The ranks of all observations are distributed according to the probability density function

$$g({}_nZ_j) = \frac{n!}{(j-1)!(n-j)!}\,{}_nZ_j^{j-1}(1 - {}_nZ_j)^{n-j} \quad (8)$$

Integration of Equation 8 gives the cumulative distribution function of ${}_nZ_j$ which is

$$G({}_nZ_j) = 1 - (1 - {}_nZ_j)^n - n\,{}_nZ_j(1 - {}_nZ_j)^{n-1} - \frac{n^{(2)}}{2!} \times {}_nZ_j^2(1 - {}_nZ_j)^{n-2} - \cdots - \frac{n^{j-1}}{(j-1)!} \times {}_nZ^{j-1}(1 - {}_nZ_j)^{n-j+1} \quad (9)$$

[NB $n^{(2)} = n(n-1)$ $n^{(3)} = n(n-1)(n-2)$ $n^{(j-1)} = n(n-1)(n-2)\ \ldots\ (n-j+2)$].

The median value of ${}_nZ_j$ is found by putting $G({}_nZ_j) = \frac{1}{2}$ and solving for ${}_nZ_j$.

TABLE I Estimates of 5% and 95% ranks for the jth observation in a sample size of $n = 50$

j	5% rank	95% rank	j	5% rank	95% rank
1	0.0010	0.0582	26	0.3954	0.6238
2	0.0072	0.0914	27	0.4148	0.6427
3	0.0166	0.1206	28	0.4343	0.6615
4	0.0278	0.1478	29	0.4540	0.6802
5	0.0402	0.1738	30	0.4739	0.6986
6	0.0536	0.1988	31	0.4940	0.7169
7	0.0676	0.2232	32	0.5142	0.7349
8	0.0822	0.2469	33	0.5347	0.7528
9	0.0972	0.2702	34	0.5554	0.7705
10	0.1127	0.2931	35	0.5763	0.7879
11	0.1286	0.3156	36	0.5974	0.8051
12	0.1447	0.3378	37	0.6187	0.8221
13	0.1612	0.3597	38	0.6403	0.8388
14	0.1779	0.3813	39	0.6622	0.8553
15	0.1949	0.4026	40	0.6844	0.8714
16	0.2121	0.4237	41	0.7069	0.8873
17	0.2295	0.4446	42	0.7298	0.9028
18	0.2472	0.4653	43	0.7531	0.9178
19	0.2651	0.4858	44	0.7768	0.9324
20	0.2831	0.5060	45	0.8012	0.9464
21	0.3014	0.5261	46	0.8262	0.9598
22	0.3198	0.5460	47	0.8522	0.9722
23	0.3385	0.5657	48	0.8794	0.9834
24	0.3573	0.5852	49	0.9086	0.9928
25	0.3762	0.6046	50	0.9418	0.9990

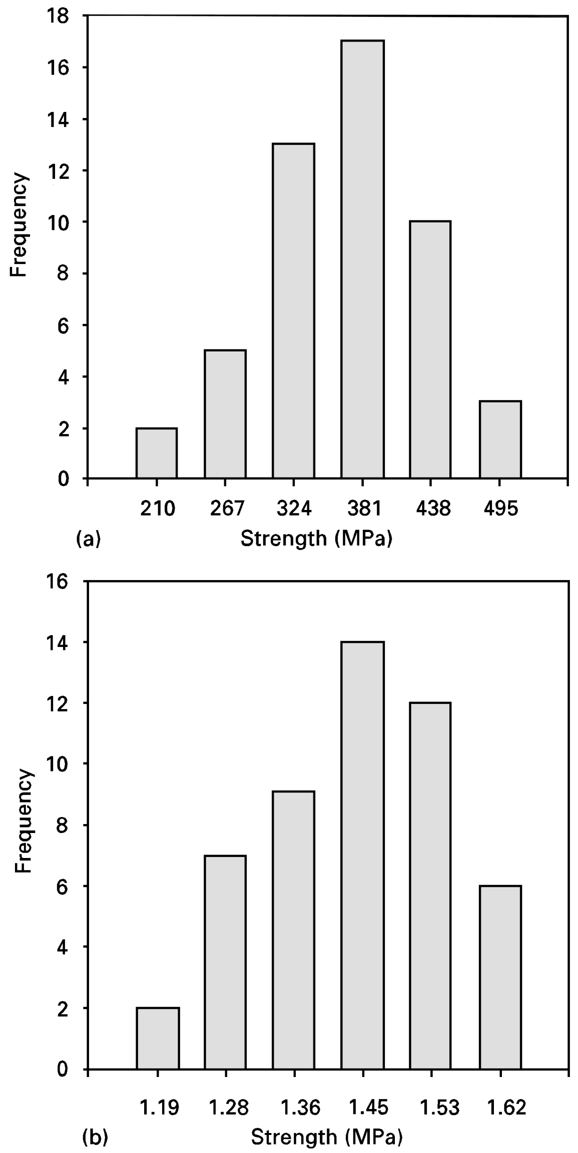

Figure 1 Histograms shown for (a) chemically toughened glass and (b) phosphate-bonded investment based on 50 measurements for each material and data divided into six bins.

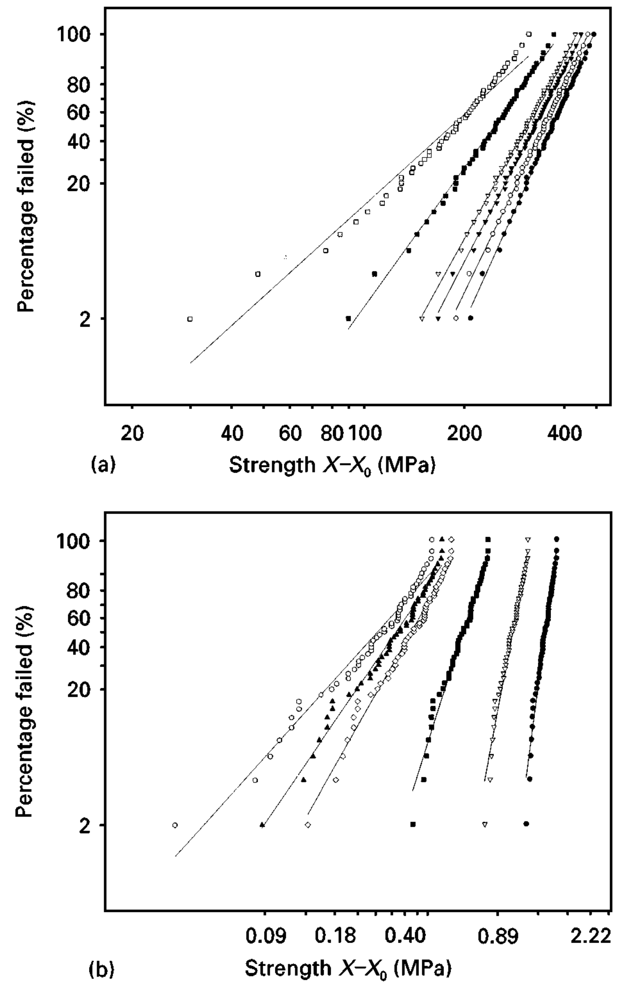

Figure 2 Weibull plots shown for (a) chemically toughened glass and (b) phosphate-bonded investment showing the modified data plotted as a function of the lower bound of strength, x_0. (a) $x_0 =$ (●) 0, (○) 20, (▼) 42.2, (▽) 60, (■) 120, (□) 180. (b) $x_0 =$ (●) 0, (▽) 0.4, (■) 0.8, (◇) 1.048, (▲) 1.1, (○) 1.5.

Thus

$$1 - (1 - {}_nZ_j)^n - n\,{}_nZ_j(1 - {}_nZ_j)^{n-1} - \frac{n^{(2)}}{2!}\,{}_nZ_j^2 \times (1 - {}_nZ_j)^{n-2} - \cdots - \frac{n^{(j-1)}}{(j-1)!} \times {}_nZ_j^{j-1}(1 - {}_nZ_j)^{n-j+1} = \tfrac{1}{2} \quad (10)$$

To determine the rank below which 5% of all jth observations in n are expected Equation 9 would become

$$1 - (1 - {}_nZ_j)^n - n\,{}_nZ_j(1 - {}_nZ_j)^{n-1} - \frac{n^{(2)}}{2!}\,{}_nZ_j^2 \times (1 - {}_nZ_j)^{n-2} - \cdots - \frac{n^{(j-1)}}{(j-1)!} \times {}_nZ_j^{j-1}(1 - {}_nZ_j)^{n-j+1} = \tfrac{5}{100} \quad (11)$$

and to determine the rank below which 95% of all jth observations in n are expected Equation 9 would

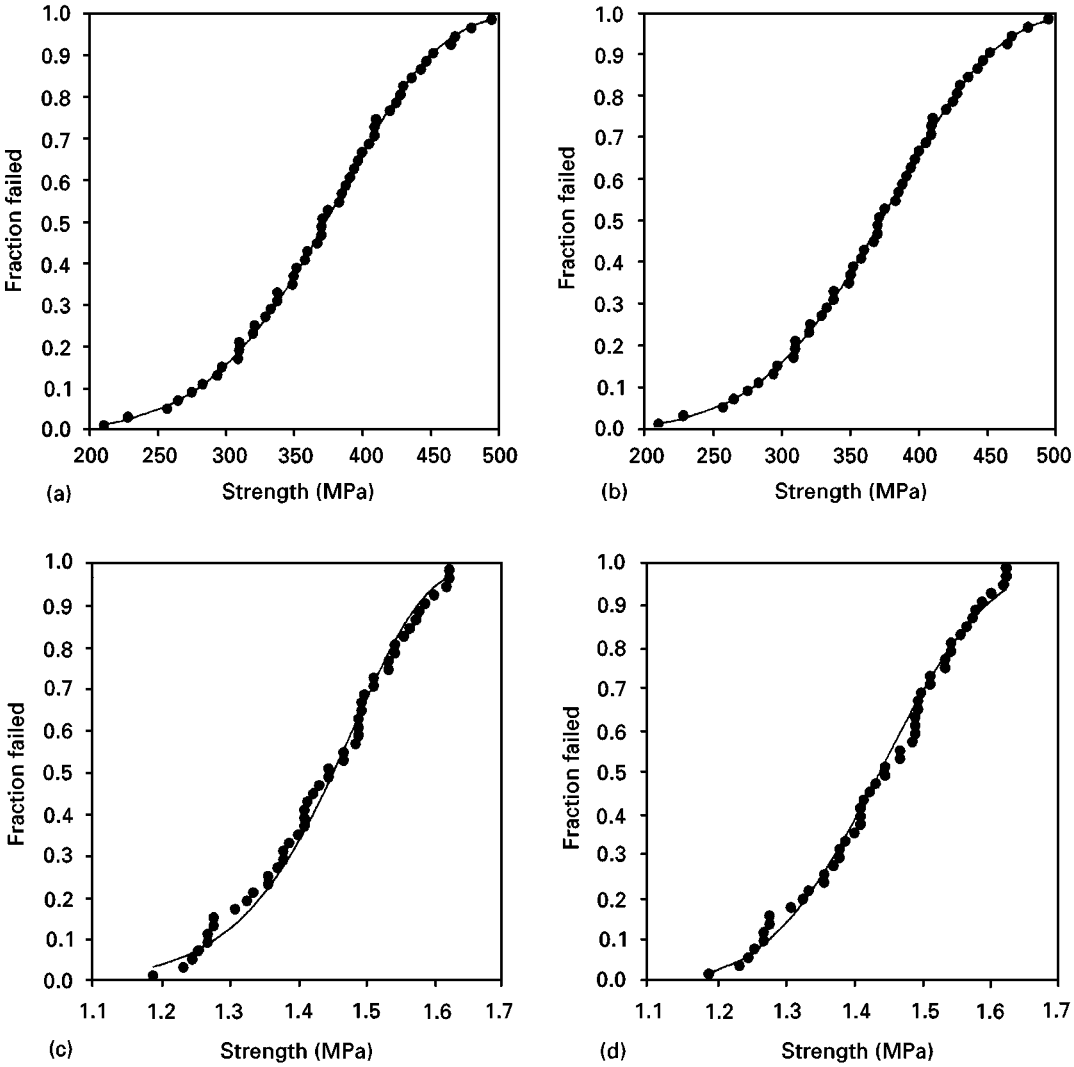

Figure 3 Measured and calculated cumulative distribution functions shown for the two materials based on the estimated values of the Weibull parameters by regression analysis. (a) Chemically toughened glass, two-parameter model: $x_0 = 0$, $\theta = 394.811$, $m = 6.436$. (b) Chemically toughened glass, three-parameter model: $x_0 = 42.2$, $\theta = 394.460$, $m = 5.611$. (c) Investment, two-parameter model: $x_0 = 0$, $\theta = 1.488$, $m = 14.74$. (d) Investment, three-parameter model: $x_0 = 1.05$, $\theta = 1.480$, $m = 3.542$.

become

$$1 - (1 - {}_nZ_j)^n - n\,{}_nZ_j(1 - {}_nZ_j)^{n-1} - \frac{n^{(2)}}{2!}\,{}_nZ_j^2 \times (1 - {}_nZ_j)^{n-2} - \cdots - \frac{n^{j-1}}{(j-1)!} \times {}_nZ_j^{j-1}(1 - {}_nZ_j)^{n-j+1} = \tfrac{95}{100} \qquad (12)$$

Thus, if $n = 50$ and $j = 2$ the 5% and 95% ranks, 0.0072 and 0.0914, respectively, are determined by solving Equations 11 and 12. The middle 90% of all second failures in 50 would be found between these values.

Tables of solutions exist for these equations but it was found necessary to generate the solutions for $n = 50$ since these were not readily available (Table I).

3. Results and discussion

The histograms for the two data sets are shown in Fig. 1 and were derived by dividing the data points between six bins in each case.

The parameters for the Weibull models based upon two and three parameters were found using the modified Weibull plots shown in Fig. 2. The two-parameter model was applied by fitting the modified data for $x_0 = 0$ and the three-parameter model by estimating the value of the lower bound of strength, x_0, that gave the best fit straight line using the least squares method.

The estimated parameters were then used to generate the cumulative distribution functions for the two and three parameter models for chemically toughened glass and phosphate-bonded investment and the results are shown in Fig. 3 along with the measured values of strength.

The cumulative distribution functions showing 90% tolerance limits for the three parameter models for chemically toughened glass and phosphate-bonded investment are shown in Fig. 4.

The χ^2 statistic was used to determine the suitability of the models to fit the measured data and histograms are shown in Fig. 5 which compares the measured data with the estimated distributions. The estimates were generated using the model parameters.

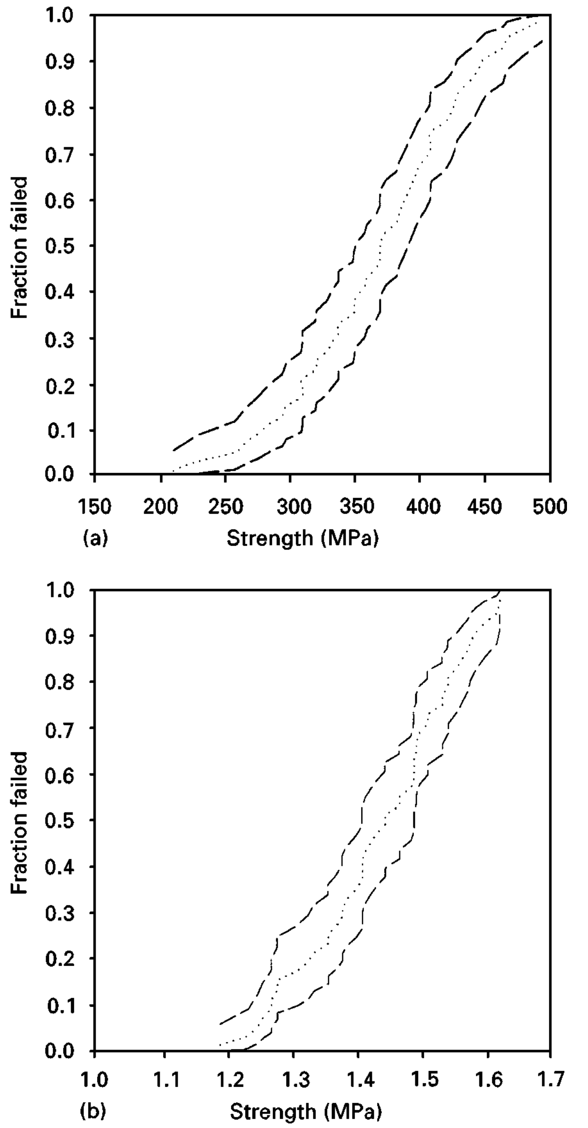

Figure 4 Ninety percent tolerance limits for the strengths of (a) chemically toughened glass and (b) phosphate-bonded investment as a function of fraction failed.

Fig. 6 is a box plot of both data sets shown on the same scale that shows the scatter associated with chemically toughened glass compared to phosphate-bonded investment. This shows that although components produced from toughened glass could be designed with higher strengths it appears that there is more uncertainty associated with the value of fracture strength for this material.

The histograms for the two materials (Fig. 1) show that both data sets are skewed, as would be expected if the Weibull distribution was fitted. It is generally known that properties such as strength and fatigue life are best represented by the Weibull distribution rather than the normal distribution especially when measured values exhibit large scatter [12].

The method that is described to fit the two- and three-parameter models has the advantage that it is straightforward and in the case of the three-parameter model allows the lower bound of strength, x_0, to be estimated. By varying, x_0, the data sets are seen to curve, first one way and then the other, about the value of x_0 that gives best fit by linear regression. The number of decimal places that are chosen for x_0 significantly changes the regression coefficient and thus the values of the Weibull modulus and characteristic life. It should be apparent that the accuracy with which strength data can be measured can influence the regression analysis significantly. However, should the data fit the model as shown for chemically toughened glass both two- and three-parameter models are observed to be in good agreement. Regression analysis is only one of several methods that have been used to estimate the Weibull parameters and other methods such as moments method or maximum likelihood are reportedly more robust [15]. An exact solution for the dependence of standard deviation on the number of tests has been evaluated by the maximum likelihood procedure but for a known Weibull modulus, m.

Given that the magnitudes of the Weibull parameters can differ when two or three parameters are used, as shown for phosphate-bonded investment, it is necessary to determine a method that will allow the best fitting model to be chosen. χ^2, for goodness of fit has been used here to compare model distributions with measured distributions and it is clear that the three-parameter models are more appropriate in both cases (Fig. 5). This result is more important for phosphate-bonded investment than for chemically toughened glass since the Weibull modulus is so widely different for the investment material, $m = 14.74$ (two-parameter model) compared to $m = 3.54$ (three-parameter model). The ability of the three-parameter model to fit the phosphate-bonded investment data is more apparent for the lower half of the data set where the curve more faithfully follows the data points.

For design purposes lower and upper limits of strength have been determined based on ranks. Table II shows the lower and upper limits for characteristic strength θ (63.2% failed), median strength (50% failed) and B_{10} strength (10% failed). These values represent the 90% tolerances for design strength for both materials. Thus, for example, it is possible to state that there is 90% certainty that the B_{10} strength would be between 1.24 and 1.31 for phosphate-bonded investment at room temperature in four-point bending at a liquid to powder ratio of 0.12.

4. Conclusions

1. The Weibull distribution can be used to describe strength measurements for chemically toughened glass and phosphate-bonded investment.

2. The χ^2 statistic showed that the three parameter models gave better descriptions of the data sets of both materials but these results had more significance for phosphate-bonded investment since the Weibull modulus was quite different for the two models. Characteristic life on the other hand was apparently independent of model chosen.

3. Ninety percent tolerance limits for design strengths such as true characteristic strength, median strength and true B_{10} strength were calculated for both materials.

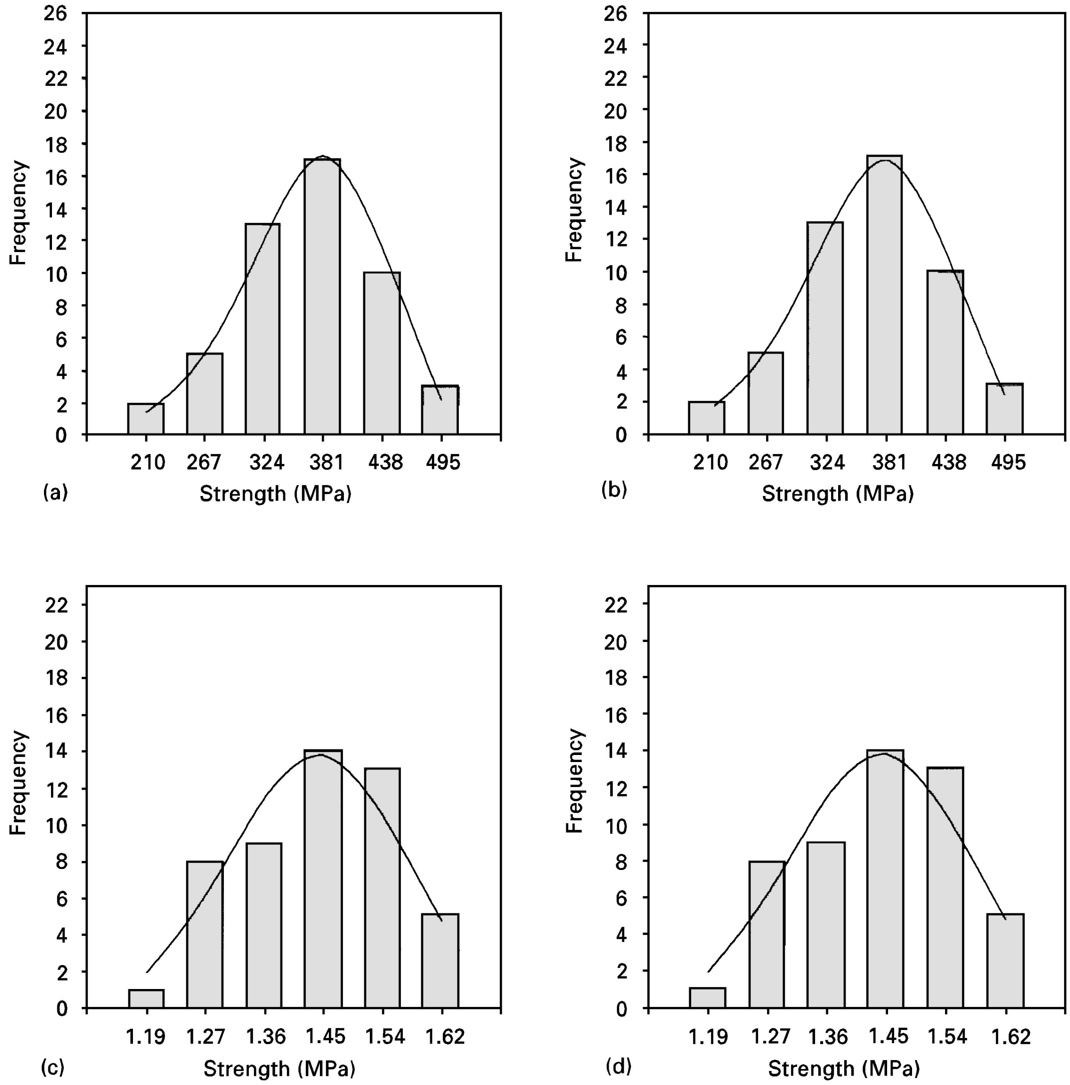

Figure 5 Histograms showing comparison of distributions for measured and calculated data with the corresponding value for goodness of fit, χ^2, for the specific material and model. Chemically toughened glass, (a) $\chi^2 = 0.16$ and (b) $\chi^2 = 0.11$. Phosphate-bonded investment, (c) $\chi^2 = 1.14$ and (d) $\chi^2 = 1.12$. (▤) data; (——) (a) and (c) two-parameter model and (b) and (d) three-parameter model.

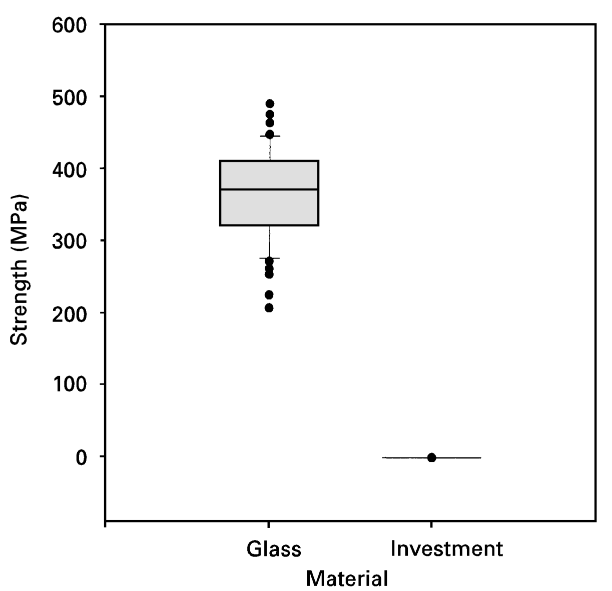

Figure 6 Box plot showing measured data points for both materials. The plot shows the 10th, 25th, 50th, 75th and 90th percentiles as lines on a bar centred about the mean, and the 5th and 95th percentiles as error bars. The data points beyond the 5th and 95th percentiles are also displayed.

TABLE II Confidence intervals (90%) for various measures of strength for the two data sets of chemically toughened glass and phosphate-bonded investment

Design strengths (MPa)	Chemically toughened glass		Phosphate-bonded investment	
	5% rank	95% rank	5% rank	95% rank
θ, true characteristic strength	372.7	409.6	1.45	1.53
True median strength	351.4	391.9	1.41	1.49
True B_{10} strength	236.9	309.2	1.24	1.31

References

1. R. V. CURTIS, A. S. JUSZCZYK, J. D. WALTER and B. SWALE, in Proceedings of the Eighth World Conference on Titanium, Birmingham, October 1995, edited by P. A. Blenkinsop, W. J. Evans and H. M. Flower (The Institute of Materials, London, 1996) p. 1784.
2. M. ITO, O. OKUNO, M. AI and I. MIURA, *Kokubyo Gakkai Zasshi*: *J. Stomatolog. Soc. Jpn* **56** (1989) 429.

3. O. OKUNO, T. NAKANO, H. HAMANAKA, I. MIURA, M. ITO, M. AI and M. OKADA, *Shika Zairyo, Kikai.* **8** (1989) 129.
4. H. MURAKAMI, *Aichi Gakuin Daigaku Shigakkai Shi, J. Dental Sci.* **27** (1989) 61.
5. S. ONODERA, T. HOSOI, M. MORITO, E. MIWA, Y. OKIKURA, F. TSUCHIDA, Y. MÜDA and M. MORI, *Nippon Hotetsu Shika Gakkai Zasshi.* **33** (1989) 1381.
6. N. WAKABAYASHI, M. AI and O. OKUNO, *Shika Zairyo, Kikai.* **9** (1990) 271.
7. M. KATO, T. SONODA, M. YAMADA, K. NAGANUMA and H. MURAKAMI, *Nagoya Ind. Technolog. Res. Inst.* **39** (1990) 318.
8. M. KATO and H. MURAKAMI, *ibid.* **39** (1990) 309.
9. M. OKADA and H. MITSUYA, *The Sumitomo Search* **43** (1990) 15.
10. N. WAKABAYASHI, *Kokubyo Gakkai Zasshi, J Stomatolog. Soc. Jpn* **59** (1992) 48–67.
11. J. MENCIK "Strength and fracture of glass and ceramics" (Elsevier, New York, 1992) p. 198.
12. C. LIPSON and N. J. SHETH, in "Statistical design and analysis of engineering experiments" edited by B. J. Clark and J. W. Maisel (McGraw-Hill, New York, 1973) p. 36.
13. L. G. JOHNSON, *Ind. Math.* **2** (1951) 1.
14. N. ORLOVSKAJA, H. PETERLIK, M. MARCZEWSKI and K. KROMP, *J. Mater. Sci.* **32** (1997) 1903.

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